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Some results connected to Dedekind's Zeta functions

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On behalf of the 150th anniversary of R. Dedekind's birthday on 6 October 1831 we present three results in which Dedekind zeta functions and Artin L-functions are involved. In a) we recall the main contents of our paper [4]. In b) the construction of a holomorphic Artin L-function $L(s,\chi,K/k)$ is given for which χ is a non-monomial irreducible character of a certain group G. In c) we regard the group GL(2,3) and we construct some relations among Artin L-functions derived from it.

The notation and conventions are those of A. Fröhlich's book "Algebraic Number Fields", Acad. Press, London, 1977, of B. Huppert's book "Endliche Gruppen, I", Springer Verlag, Berlin–Heidelberg, 1967, and of I.M. Isaacs' book "Character theory of finite groups", Acad. Press, London, 1976. References to these sources are written [F], [H], [I] respectively. Representation theory of groups is done over the complex number field.

a) (1977, [F], pages 649–662; see also [4] in which the explicit proofs are given)

Let L be an algebraic number field (= extension field of finite degree of the rational field $\mathbb{Q}$). Let

$$\zeta_L(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s},$$

defined in the domain $\mathrm{Re}(s) > 1$, the summation being extended over all integral ideals of L. The function $\zeta_L(s)$, s complex variable, is called the zeta function of Dedekind. The zeta function $\zeta_L(s)$ converges absolutely and uniformly in the domain $\mathrm{Re}(s) \geqq 1+\delta$, any $\delta > 0$. It was E. Hecke, who found the functional equation for $\zeta_L(s)$ and he proved that $\zeta_L(s)$ has an analytic continuation over the whole complex plane, with the sole exception at the point $s=1$, where there is a simple pole. Notice that $\zeta_{\mathbb{Q}}(s)$ is just the ordinary Riemann zeta function.

In [1], E. Artin poses the following conjecture.

CONJECTURE. Let M be a finite extension of L. Then $\zeta_M(s)/\zeta_L(s)$ is holomorphic in the whole complex plane.

In my 1977-survey paper in [F] I gave some cases for which the conjecture becomes a theorem. The main result was the following

THEOREM. Let $\Omega \supseteq M \supseteq L$ be fields with $[\Omega : L] < \infty$. Suppose that Ω/L is a galois extension with $\mathrm{Gal}(\Omega/L)$ a solvable group. Then $\zeta_M(s)/\zeta_L(s)$ is a holomorphic function.

The proof is based on the following proposition proved in [4].

PROPOSITION. Let $X \supseteq Y \supsetneqq L$ be fields with $[X : L] < \infty$ and assume that X/L is a galois extension with galois group $G = \mathrm{Gal}(X/L)$. Furthermore assume that $H = \mathrm{Gal}(X/Y)$ is a maximal subgroup of G and that $G = HA$, with A some normal abelian subgroup of G with $H \cap A = \{1\}$. Let ψ be the principal character of H. Then

$$\psi^G = e + \sum_{i=1}^{t} e_i,$$

where e is the principal character of G and where the e_i are non-principal monomial irreducible characters of G; here $t \geqq 1$ and $e_j \neq e_m$ if $j \neq m$. Therefore, by Artin's formalism,

$$\zeta_Y(s)/\zeta_L(s) = \prod_{i=1}^{t} L(s, e_i, X/L)$$

is a holomorphic function.

b) Let r be a prime number with $r \equiv 3 \pmod 4$. Let E be an extra special r-group of order r^3 and of exponent r, that is,

$$E = \langle s,t,z \mid s^r = t^r = z^r = [s,z] = [t,z] = 1,\ z = [s,t] \rangle.$$

The quaternion group Q of order 8 acts on E as follows. Let

$$Q = \langle p,q \mid p^4 = 1,\ p^2 = q^2,\ q^{-1}pq = p^{-1} \rangle.$$

It is well known that every element of any finite field is a sum of two squares in that field. Thus we can choose

$\alpha, \beta \in \mathbb{Z}$ with $\alpha^2 + \beta^2 = -1 \pmod r$. Set $s^p = s^\alpha t^\beta$, $t^p = s^\beta t^{-\alpha}$, $s^q = t$, $t^q = s^{-1}$.

Let G be the relative holomorph of E with Q. The concept is clear. It follows that $z^p = z$, whence $Z(E) = Z(G)$. Let $\chi \in \mathrm{Irr}\, E$ be an irreducible character of E with $\chi(1) = r$. Let $\chi(z) = r\omega$, ω some r^{th}-root of unity with $\omega \neq 1$. By theorem V.16.14 of [H], $\chi \in \mathrm{Irr}\, E$ is the unique irreducible character of E for which $(\chi_{Z(E)}, \lambda) \geqq 1$, with $\lambda(z) = \omega$, $\lambda \in \mathrm{Irr}\, Z(E)$. Define $\mu \in \mathrm{Irr} \langle s,z \rangle$ by $\mu(s) = 1$, $\mu(z) = \lambda(z) = \omega$. Hence μ is a 1-dimensional character with $\mathrm{Ker}\, \mu = \langle s \rangle$. The group $H = \langle s,z,p^2 \rangle$ has order $2r^2$ and H admits precisely two distinct 1-dimensional characters ξ_1 and ξ_2 with $\xi_1|_{\langle s,z \rangle} = \xi_2|_{\langle s,z \rangle} = \mu$. By theorem V.16.14 of [H] again, we have $\mu^E = \chi$. The character χ can be extended to a character $\bar{\chi} \in \mathrm{Irr}\, G$, by theorem V.17.12 of [H]. Hence we have

$$\chi^G = \sum_{i=1}^{5} \psi_i(1)\, (\bar{\chi} \otimes \psi_i),$$

where $\psi_1, \psi_2, \psi_3, \psi_4$ are the 1-dimensional characters of Q, regarded as characters of G/E, inflated to G; $\psi_5 \in \mathrm{Irr}\, G$ is the 2-dimensional character of G/E, regarded as character of G. The five distinct characters $\bar{\chi} \otimes \psi_1, \ldots, \bar{\chi} \otimes \psi_5$ are precisely all irreducible characters of G lying over χ. Next remark, that $(\bar{\chi} \otimes \psi_i)_{E\langle p^2 \rangle}$ gives rise to one irreducible character $\eta \in \mathrm{Irr}\, E<p^2>$ for any $i = 1,2,3,4$ with $\eta_E = \chi$. Hence $\chi^{E<p^2>} = \eta + (\eta \otimes \delta)$, where $\eta \otimes \delta \in \mathrm{Irr}\, E<p^2>$, $\eta \neq \eta \otimes \delta$, $\delta(f) = 1$ for all $f \in E$, $\delta(f) = -1$ for all $f \in E<p^2> - E$. Let $\varphi \in \mathrm{Irr}\, G$ have $\eta \otimes \delta$ in its restriction to $E<p^2>$. Hence χ is contained in φ_E. Therefore $\varphi \in \{\bar\chi \otimes \psi_i \mid i = 1,2,3,4,5\}$ and we see that $\varphi = \chi \otimes \psi_5$. It follows that

$$((\bar\chi \otimes \psi_5)_{E<p^2>}, \eta) = (\bar\chi \otimes \psi_5, \eta^G) = (\bar\chi \otimes \psi_5, \sum_{i=1}^{4} (\bar\chi \otimes \psi_i)) = 0.$$

Thus $(\bar\chi \otimes \psi_5)_{E<p^2>} = 2(\eta \otimes \delta)$. By Frobenius' reciprocity for groups we have now $(\eta \otimes \delta)^G = 2(\bar\chi \otimes \psi_5) + \ldots$, but also $(\eta \otimes \delta)^G(1) = 4((\eta \otimes \delta)(1)) = 2((\bar\chi \otimes \psi_5)(1))$. Hence $(\eta \otimes \delta)^G = 2(\bar\chi \otimes \psi_5)$. Now $\mu^{E<p^2>} = \chi^{E<p^2>} = \eta + (\eta \otimes \delta) = (\mu^H)^{E<p^2>} = \xi_1{}^{E<p^2>} + \xi_2{}^{E<p^2>}$. As $\eta \in \mathrm{Irr}\, E<p^2>$ and $\eta \otimes \delta \in \mathrm{Irr}\, E<p^2>$ we can choose $\xi_2{}^{E<p^2>} = \eta \otimes \delta$. Hence $\xi_2{}^G = (\eta \otimes \delta)^G = 2(\bar\chi \otimes \psi_5)$.

Let G be the galois group of the galois extension M/L. Let Ω be the invariant field for H. Then $L(s,\xi_2,M/\Omega) = L(s,\xi_2{}^G,M/L) = L(s,2(\bar\chi \otimes \psi_5),M/L) = (L(s,\bar\chi \otimes \psi_5,M/L))^2$ is a holomorphic function as $L(s,\xi_2,M/\Omega)$ is a Hecke–Dirichlet L-function because of $\xi_2(1) = 1$. A result of R. Brauer yields that $L(s,\bar\chi \otimes \psi_5,M/L)$ is a meromorphic function, see Theorem V.19.3 of [H]. Hence $L(s,\bar\chi \otimes \psi_5,M/L)$ itself is now holomorphic. We have $(\bar\chi \otimes \psi_5)(1) = 2r$. We shall show that G does not contain subgroups of index 2r, that is, $\bar\chi \otimes \psi_5$ is not a monomial character. Indeed, let $U \subset G$ be a subgroup of G with index 2r. Since E is a normal subgroup of G, we have $EU/E \cong U/(E \cap U)$. If $G = UE$, then $2r = [G:U] = [UE:U] = [E:(E \cap U)]$ divides $|E| = r^3$, a contradiction. Therefore $G \neq UE$, i.e. $[UE:U] = [E:(E \cap U)]$ divides $|E| = r^3$. Since E is the Sylow r-subgroup of G, $UE \neq U$. Thus $[G:UE] = 2$. Now UE is equal to $E<p>$, $E<q>$ or $E<pq>$ and this exhaust all possibilities. However, $<p>$, $<q>$ and $<pq>$ operate irreducibly by conjugation on E/Z(E), as $p^2 = q^2 = (pq)^2$ inverts any element of E/Z(E) and as $r \equiv 3 \pmod 4$. Therefore such a group U does not exist.

Finally we note that the case $r = 3$ was found by J.G. Thompson (letter to J.P. Serre, dated July 27, 1974).

c) We refer to [3]. Let $G = GL(2,3)$. The group G is solvable and $|G| = 48$. According to I. Schur (1907) G has the following character table.

representant of conjugacy class	number of el. in conj. class	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1	1	2	2	2	3	3	4
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1	1	1	−2	−2	2	3	3	−4
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	6	1	1	0	0	2	−1	−1	0
$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	8	1	1	1	1	−1	0	0	−1

representant of conjugacy class	number of el. in conj. class	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8
$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	8	1	1	-1	-1	-1	0	0	1
$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$	12	1	-1	0	0	0	1	-1	0
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	6	1	-1	$i\sqrt{2}$	$-i\sqrt{2}$	0	-1	1	0
$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$	6	1	-1	$-i\sqrt{2}$	$i\sqrt{2}$	0	-1	1	0

Let $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Hence $|<b>| = 6$. Let $\alpha \in \mathrm{Irr} <b>$ with $\alpha(1) = \alpha(b^2) = \alpha(b^4) = 1$, $\alpha(b) = \alpha(b^3) = \alpha(b^5) = -1$. Notice that $b^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. It follows that $\alpha^G = 2\psi_8$. Let $e = e^{2\pi i/6}$. Define $\gamma(b^t) = e^t$ for $1 \leqq t \leqq 6$ and write $\bar{\gamma}(b^t) = \gamma(b^{-t})$; whence $\gamma, \bar{\gamma} \in \mathrm{Irr} <b>$. We have

$$\gamma^G = \psi_3 + \psi_4 + \psi_8 = \bar{\gamma}^G.$$

Therefore $2(\psi_3 + \psi_4) = \gamma^G + \bar{\gamma}^G - \alpha^G$.

Next, let M/S be a galois extension with $G \cong \mathrm{Gal}(M/S)$. Then it follows that

$$(L(s, \psi_3 + \psi_4, M/S))^2 = \frac{L(s, \gamma + \bar{\gamma}, M/U)}{L(s, \alpha, M/U)},$$

where U is the invariant field of the group $<b>$, considered as subgroup of Gal(M/S). The L-functions $L(s,\alpha,M/U)$ and $L(s,\gamma+\bar{\gamma},M/U)$ are holomorphic, following Hecke. So not only for $\mathrm{Re}(s) \geqq 1$, but also for any $s \in \mathbb{C}$ $(L(s,\psi_3+\psi_4,M/S))^2$ is defined by the last formula. We know by a result of R. Brauer that $L(s,\psi_3+\psi_4,M/S)$ is meromorphic, and it follows also that $L(s,\psi_8,M/S)$ is holomorphic, as ψ_8 is monomial. We drop the symbols M,S,U for the moment.

Now observe that $\alpha(b^3) = \gamma(b^3) = \bar{\gamma}(b^3) = -1$. Hence the functional equation for Artin L-functions as given in [2], page 306, gives the following formulae.

$$L(1-s,\gamma) = W(\gamma)\,(N\mathfrak{F}(\gamma,M/U))^{s-\frac{1}{2}}\,A(s,\gamma)\,L(s,\bar{\gamma}) \qquad \text{x)}$$

$$L(1-s,\alpha) = W(\alpha)\,(N\mathfrak{F}(\alpha,M/U))^{s-\frac{1}{2}}\,A(s,\alpha)\,L(s,\bar{\alpha}) \qquad \text{xx)}$$

Just as $\alpha(b^3) = \gamma(b^3)$, it follows that $A(s,\alpha) = A(s,\gamma)$. The reader is referred to the explicit formulae for $A(s,\gamma)$ (and for $A(s,\alpha)$) in [2]. We have $\alpha = \bar{\alpha}$ and $L(s,\gamma) = L(s,\bar{\gamma})$ by $\gamma^G = \bar{\gamma}^G$. Keep in mind that both $L(s,\alpha)$ and $L(s,\gamma)$ are zero-point-free in $\mathrm{Re}(s) \geqq 1$. It follows that

$$\begin{aligned} &L(s,\alpha)\,A(s,\alpha)\,W(\alpha)\,(N\mathfrak{F}(\alpha,M/U))^{s-\frac{1}{2}}\,L(1-s,\gamma) = \\ &\qquad = W(\gamma)\,(N\mathfrak{F}(\gamma,M/U))^{s-\frac{1}{2}}\,A(s,\gamma)\,L(s,\bar{\gamma})\,L(1-s,\alpha). \end{aligned}$$

As $A(s,\alpha)$ = product of cos-, sin-, and Γ-functions, we see that

$$\begin{aligned} &L(s,\alpha)\,W(\alpha)\,(N\mathfrak{F}(\alpha,M/U))^{s-\frac{1}{2}}\,L(1-s,\gamma) = \\ &\qquad = W(\gamma)\,(N\mathfrak{F}(\gamma,M/U))^{s-\frac{1}{2}}\,L(s,\bar{\gamma})\,L(1-s,\alpha) \end{aligned} \qquad \genfrac{}{}{0pt}{}{\text{x}}{\text{xx}})$$

for those s for which $A(s,\alpha) \neq 0$. However, if $A(s_0,\alpha)=0$, then $L(1-s_0,\alpha)=0$ as $L(s,\alpha)$ is analytic for all $s\in\mathbb{C}$, following formula xx). The same holds for formula x). Hence formula $^{\,x}_{xx}$) holds for all $s\in\mathbb{C}$. Notice that $L(s,\bar{\gamma}) = L(s,\gamma) = L(s,\psi_3+\psi_4)\, L(s,\psi_8)$ for all $s\in\mathbb{C}$. Further $L(s,\alpha) = (L(s,\psi_8))^2$ for any $s\in\mathbb{C}$. Hence we find that

$$L(s,\psi_8)\, L(1-s,\psi_8)\, \{L(s,\psi_8)\, W(\alpha)\, (N\mathfrak{F}(\alpha,M/U))^{s-\frac{1}{2}}\, L(1-s,\psi_3+\psi_4) - W(\gamma)\, (N\mathfrak{F}(\gamma,M/U))^{s-\frac{1}{2}}\, L(s,\psi_3+\psi_4)\, L(1-s,\psi_8)\} = 0.$$

Write in short $L(s,\psi_8)\, L(1-s,\psi_8)\, A(s)=0$. Now, if $L(s_1,\psi_8)=0$ for $\frac{1}{2}\leqq \mathrm{Re}(s_1)<1$, then also $L(1-s_1,\psi_8)=0$ by the functional equation. Hence $A(s_1)=0$. For $|\mathrm{Re}(s_i)|\geqq 1$ we can only have $L(1-s_i,\psi_8)=0$ for $s_i=1,2,3,\ldots$. For those points s_i it follows that $L(1-s_i,\alpha)=0$. Look at $^{\,x}_{xx}$). Then $L(1-s_i,\gamma)=0$ and by $^{\,x}_{xx}$) again, the order of the zero s_i in $L(1-s_i,\gamma)$ is equal to the order of the zero s_i in $L(1-s_i,\alpha)$. Therefore

$$(L(1-s_i,\psi_3+\psi_4))^2 = \frac{(L(1-s_i,\gamma))^2}{L(1-s_i,\alpha)} = 0.$$

The final result ist that $A(s)=0$ for all $s\in\mathbb{C}$, as now $L(1-s_i,\psi_3+\psi_4)=0$.

Next we look at the several Artin root numbers involved here. By the corollary on page 18 of [F] we have

$$W(\gamma) = W(\psi_3+\psi_4+\psi_8) = W(\psi_3)\, W(\psi_4)\, W(\psi_8) \text{ and}$$
$$W(\alpha) = W(2\psi_8) = (W(\psi_8))^2.$$

The Artin root numbers are complex roots of unity and it holds that $W(\psi_4)$ is the complex conjugate to $W(\psi_3)$ as $\psi_4=\overline{\psi_3}$. Hence $W(\psi_3)\, W(\psi_4)=1$ and so $W(\gamma) = W(\psi_8)$. The character ψ_8 is afforded by a real representation of $GL(2,3)$. Thus we can apply a theorem of A. Fröhlich and J. Querut, see [F] page 124, that says that in such a situation $W(\psi_8)=1$. Hence $W(\gamma)=W(\psi_8)=1=(W(\psi_8))^2=W(\alpha)$.

Therefore we have proved that

$$L(s,\psi_8)\, L(1-s,\psi_3+\psi_4)\, (N\mathfrak{F}(\alpha,M/U))^{s-\frac{1}{2}} = L(1-s,\psi_8)\, L(s,\psi_3+\psi_4)\, (N\mathfrak{F}(\gamma,M/U))^{s-\frac{1}{2}}.$$

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